Derivations of operator algebras

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1. Introduction

This paper is concerned with results describing the nature of derivations of operator algebras — especially, derivations of von Neumann algebras. Neglecting convergence questions, which can be dealt with effectively in this case, the exponential of a derivation will be an automorphism. The adjoint-preserving automorphisms α of a C^*-algebra $\mathfrak{A}$ acting on a Hilbert spece $\mathcal{H}$ cannot, in general, be implemented by a unitary operator U on $\mathcal{H}$ (as $\alpha(A) = U^*AU$, for each A in $\mathfrak{A}$). It follows from [9; Cor. 2.3.1] that α extends to the weak closure $\mathfrak{A}^-$ of $\mathfrak{A}$ (for $\mathcal{H}$ separable) if and only if α preserves the null ideal of the representation of $\mathfrak{A}$ involved. The work of Murray and von Neumann [17; Th. XI, 18; Th. X] indicates that automorphisms of von Neumann algebras with no part of type III tend to be *spatial* (i.e., implemented by a unitary transformation). Griffin's results [7, 8] extend this to the type III situation. Kaplansky [12] has noted that automorphisms of type I von Neumann algebras which leave the center elementwise fixed are inner. It is well known that automorphisms of factors not of type I will not usually be inner. In fact, N. Suzuki [25] shows that each countable group is isomorphic to a group of outer automorphisms of the hyperfinite II_1 factor.

By analogy with his type I automorphism results, Kaplansky [13; Th. 9] establishes that each derivation of a type I von Neumann algebra is inner. He proceeds from the result of I. M. Singer that each derivation of a commutative C^*-algebra is 0. Singer and Wermer [24] proved analogous results for commutative Banach algebras. An extension of Singer's result (cf. Theorem 2) establishes that derivations of a C^*-algebra annihilate the center which accounts for the fact that Kaplansky's result (which plays a key role in our work) does not require the normalization on the center present in the automorphism case. Kaplansky was led to conjecture that each derivation of a C^*-algebra is continuous. This was proved by S. Sakai [21]. Using these results, P. Miles [26] shows that each derivation of a C^*-algebra is induced by an operator in the weak closure in some faithful representation of the algebra.

That represents the state of our knowledge about derivations of von Neumann algebras not of type I (cf. [2; p. 257]). The relation of derivations to

* Research conducted with the support of NSF GP 1604 and in part with the support of ONR contract NR 043-325 and NSF GP 4059.

automorphisms and our relatively complete information about automorphisms of operator algebras makes numerous "informed" guesses available. By analogy with the case of automorphisms, we say that a derivation δ of a C^*-algebra $\mathfrak{A}$ acting on $\mathcal{H}$ is *spatial* when there is a bounded operator B on $\mathcal{H}$ such that $\delta(A) = BA - AB$ (=ad $B(A)$), for each A in $\mathfrak{A}$. If B can be chosen in $\mathfrak{A}$, we say that δ is *inner*. The guesses would be that there are non-spatial derivations of C^*-algebras and non-inner derivations of von Neumann algebras. Our results establish the negation of the first guess and indicate rather strongly that the negation of the second holds. In particular we show (cf. Theorem 7) that each derivation of a hyperfinite von Neumann algebra is inner (cf. this with N. Suzuki's results quoted above). It should also be noted that certain factors of type III fall within the scope of this assertion [20; § 7, 19; p. 95].

2. Preliminary results

We say that a state ρ of a C^*-algebra $\mathfrak{A}$ is *definite* [11; p. 398] on the self-adjoint operator A in $\mathfrak{A}$ when $\rho(A^2) = \rho(A)^2$. In this case, ρ is multiplicative on the C^*-subalgebra of $\mathfrak{A}$ generated by A. The following lemma is a combination of Singer's argument that derivations of commutative C^*-algebras are 0 and results [10; Lemma] on the multiplicative properties of definite states.

LEMMA 1. *If δ is a derivation of the C^*-algebra $\mathfrak{A}$ and ρ is definite on A in $\mathfrak{A}$, then $\rho(\delta(A)) = 0$.*

PROOF. Note that $\delta(I) = \delta(I^2) = 2\delta(I)$, so that $\delta(I) = 0$. Thus $\delta(A) = \delta(A - \rho(A)I)$; and we may assume $\rho(A) = 0$. In this case, $0 = \rho(A^+) = \rho(A^-)$, where $A = A^+ - A^-$, A^+ and A^- are the "positive" and "negative" parts of A; for $A^+A = A^{+2}$, so that $0 = \rho(A^+)\rho(A) = \rho(A^+A) = \rho(A^{+2}) = \rho(A^+)^2$. Since $\delta(A) = \delta(A^+) - \delta(A^-)$, it will suffice to show that $\rho(\delta(A^+)) = \rho(\delta(A^-)) = 0$. We may assume $A > 0$ and $\rho(A) = 0$. Let $B = A^{1/2}$. Then $\rho(B) = 0$. Hence $\rho(\delta(A)) = \rho[\delta(B)B] + \rho[B\delta(B)] = \rho[\delta(B)]\rho(B) + \rho(B)\rho[\delta(B)] = 0$, from [10; Lemma].

The substance of the foregoing lemma is that each derivation of a C^*-algebra maps each self-adjoint operator in the algebra onto an operator that has 0 diagonal relative to a diagonalization which diagonalizes A.

THEOREM 2. *Each derivation of a C^*-algebra annihilates its center.*

PROOF. Let δ be a derivation of the C^*-algebra $\mathfrak{A}$ with center $\mathcal{C}$. Let ρ be a pure state of $\mathfrak{A}$, and C an element of $\mathcal{C}$. The representation of $\mathfrak{A}$ associated with ρ is irreducible [23] and therefore maps $\mathcal{C}$ into scalars. Together with the Schwarz inequality, this yields that ρ is multiplicative on $\mathcal{C}$. From the preceding lemma, $\rho(\delta(C)) = 0$. Since the pure states of $\mathfrak{A}$ separate $\mathfrak{A}$, $\delta(C) = 0$.

LEMMA 3. *If δ is a derivation of the C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}$, then δ has a unique ultra weakly continuous extension which is a derivation of $\mathfrak{A}^-$.*

PROOF. We show that for each x, y in $\mathcal{H}$, $\omega_{x,y} \circ \delta$ is strongly continuous at 0 on $\mathfrak{S}_1^+$, the positive operators in the unit ball $\mathfrak{S}_1$ of $\mathfrak{A}$. Now

$$A \longrightarrow ([A\delta(A) + \delta(A)A]x, y) \qquad (=(\delta(A^2)x, y))$$

is strongly continuous at 0 on $\mathfrak{S}_{1*}$, the set of self-adjoint operators in the unit ball of $\mathfrak{A}$, since $|([A\delta(A) + \delta(A)A]x, y)| \leqq \|\delta\| (\|Ax\| \|y\| + \|x\| \|Ay\|)$, where $\|\delta\| < \infty$ by Sakai's theorem [21]. Moreover, $A \rightarrow A^{1/2}$ is strongly continuous at 0 on positive operators, since $\|A^{1/2}x\|^2 = |(Ax, x)| \leqq \|Ax\| \cdot \|x\|$. Thus $A \rightarrow A^{1/2} \rightarrow (\delta(A)x, y)$ is strongly continuous at 0 on $\mathfrak{S}_1^+$.

We note next that δ is weakly continuous on $\mathfrak{S}_1$ to $\mathfrak{A}$ in the weak operator topology. Since $Ax = A^+x - A^-x$ with A^+x and A^-x orthogonal, $\|A^+x\| \leqq \|Ax\|$ and $\|A^-x\| \leqq \|Ax\|$; so that $A \rightarrow A^+$ and $A \rightarrow A^-$ are strongly continuous mappings on the self-adjoint operators in $\mathfrak{A}$ at 0. Thus

$$A \longrightarrow (\delta(A^+)x, y) - (\delta(A^-)x, y) = (\delta(A)x, y)$$

is strongly continuous at 0 on $\mathfrak{S}_{1*}$. By linearity this mapping is strongly continuous at 0 on $2\mathfrak{S}_{1*}$ and from this, everywhere on $\mathfrak{S}_{1*}$. Hence the inverse image of a closed convex subset of the complex numbers under $A \rightarrow (\delta(A)x, y)$ has an intersection with $\mathfrak{S}_{1*}$ which is strongly closed relative to $\mathfrak{S}_{1*}$. This intersection being convex, each weak limit point is a strong limit point [3, 15], so that it is weakly closed relative to $\mathfrak{S}_{1*}$. Since the closed convex subsets of the complex numbers form a subbase for the closed subsets, $A \rightarrow (\delta(A)x, y)$ is weakly continuous on $\mathfrak{S}_{1*}$. Now $A \rightarrow (A + A^*)/2$ and $A \rightarrow (A - A^*)/2i$ are weakly continuous mappings of $\mathfrak{S}_1$ into $\mathfrak{S}_{1*}$; so that

$$A \longrightarrow \left(\delta\left(\frac{A + A^*}{2}\right)x, y\right) + i\left(\delta\left(\frac{A - A^*}{2i}\right)x, y\right) = (\delta(A)x, y)$$

is weakly continuous on $\mathfrak{S}_1$. Thus δ is weakly continuous on $\mathfrak{S}_1$.

The linearity of δ now yields its uniform continuity relative to the weak-operator uniform structure on $\mathfrak{S}_1$. From the Kaplansky density theorem [14], $\mathfrak{S}_1^-$ is the unit ball in $\mathfrak{A}^-$, and is compact in the weak-operator topology. Thus δ has a (unique) weak-operator continuous extension to $\mathfrak{S}_1^-$, and this extension has an obvious extension $\bar{\delta}$ from $\mathfrak{S}_1^-$ to $\mathfrak{A}^-$. It is easily checked that this extension is well-defined and linear. For x in $\mathcal{H}$,

$$(A, B) \longrightarrow ([\bar{\delta}(AB) - \bar{\delta}(A)B - A\bar{\delta}(B)]x, x)$$

is strongly continuous on $\mathfrak{S}_{1*}^- \times \mathfrak{S}_{1*}^-$, by strong continuity of operator multiplication on bounded sets, weak continuity of $\bar{\bar{\delta}}$ on $\mathfrak{S}_1^-$ and boundedness of δ (hence $\bar{\delta}$). Since this mapping is 0 on $\mathfrak{S}_{1*} \times \mathfrak{S}_{1*}$, a strongly dense subset of $\mathfrak{S}_{1*}^- \times \mathfrak{S}_{1*}^-$; it is 0 on $\mathfrak{S}_{1*}^- \times \mathfrak{S}_{1*}^-$, for each x, so that $\bar{\bar{\delta}}$ is a derivation on $\mathfrak{A}^-$.

3. The main results

J. Schwartz [22] has introduced a property of von Neumann algebras which he uses to establish the existence of a third isomorphism class of factors of type II_1. We recall that $\overline{\mathrm{co}}_{\mathcal{R}}(A)$, for an arbitrary bounded operator A on $\mathcal{H}$ and $\mathcal{R}$ a von Neumann algebra, is the weak closure of $\mathrm{co}_{\mathcal{R}}(A)$, the finite convex combinations of operators UAU^* with U a unitary operator in $\mathcal{R}$. We say that A is *mobile* (relative to $\mathcal{R}$) when $\overline{\mathrm{co}}_{\mathcal{R}}(A)$ has non-null intersection with $\mathcal{R}'$; and we say that $\mathcal{R}$ is *mixing* when A is mobile for each bounded A. It is noted in [22] that each hyperfinite von Neumann algebra is mixing.

THEOREM 4. *Each derivation δ of a C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}$ is spatial. We may choose B commuting with an assigned maximal abelian subalgebra of $\mathfrak{A}'$ or with an assigned mixing von Neumann subalgebra of $\mathfrak{A}'$ so that $\delta = \mathrm{ad}\, B \mid \mathfrak{A}$.*

PROOF. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathfrak{A}'$, and let $\mathcal{P}$ be the lattice of projections in $\mathcal{A}$. From Sakai's theorem, δ is bounded on $\mathfrak{A}$; and from Lemma 3, δ has an extension $\bar{\bar{\delta}}$ to $\mathfrak{A}^-$. For $E_1, \cdots, E_n$ in $\mathcal{P}$ and $A_1, \cdots, A_n$ in $\mathfrak{A}^-$, define $\delta_1(A_1E_1 + \cdots + A_nE_n)$ to be $\bar{\bar{\delta}}(A_1)E_1 + \cdots + \bar{\bar{\delta}}(A_n)E_n$. If

$$A_1E_1 + \cdots + A_nE_n = 0\ ,$$

then there exist central operators $C_{j,k}$, $j, k = 1, 2, \cdots, n$ in $\mathfrak{A}^-$, such that $\sum_{k=1}^n C_{j,k}E_k = E_j$ and $\sum_{j=1}^n A_jC_{j,k} = 0$, from [9; Lem. 3.1.1]. Since $\bar{\bar{\delta}}$ annihilates the center of $\mathfrak{A}^-$ (Theorem 2), $0 = \sum_{j=1}^n \bar{\bar{\delta}}(A_j)C_{j,k}$; so that $\sum_j \bar{\bar{\delta}}(A_j)E_j = 0$, again from [9; Lem. 3.1.1]. Thus δ_1 as defined is single-valued. The linearity of δ_1 is clear. We note that the set $\mathfrak{A}_0$ of operators on which δ_1 is defined is an algebra and that δ_1 is a derivation. In fact, since the operators AE generate $\mathfrak{A}_0$ linearly, it suffices to check the product relation on $AEBF = ABEF$, a routine computation.

Observe next that δ_1 is bounded on $\mathfrak{A}_0$. In fact, each $A_1E_1 + \cdots + A_nE_n$ in $\mathfrak{A}_0$ can be expressed in the form $B_1F_1 + \cdots + B_kF_k$ with $B_1, \cdots, B_k$ in $\mathfrak{A}^-$ and $F_1, \cdots, F_k$ mutually orthogonal projections in $\mathcal{P}$; for if $E_1, \cdots, E_j$ are orthogonal, we replace E_{j+1} by

$$E_{j+1}(E_1 + \cdots + E_j) + E_{j+1} - E_{j+1}(E_1 + \cdots + E_j)\ .$$

Now $A_1E_1 + A_{j+1}E_{j+1}E_1 = A_1(E_1 - E_{j+1}E_1) + (A_1 + A_{j+1})E_{j+1}E_1$. In this way we replace $A_1E_1 + \cdots + A_{j+1}E_{j+1}$ by a sum in which all the projections are orthogonal. We then deal with $A_{j+2}E_{j+2}, \cdots, A_nE_n$, successively, in the same way. With $E_1, \cdots, E_n$ mutually orthogonal, $x = \sum_{j=1}^n E_jx$ and $\| x \| = 1$,

$$\| (A_1E_1 + A_2E_2 + \cdots + A_nE_n)x \|^2 = \sum_{j=1}^n \| A_jE_jx \|^2 ;$$

since $\{A_jE_jx\} = \{E_jA_jx\}$ are mutually orthogonal vectors. Now

$$\begin{aligned}\sum_{j=1}^n \| A_jE_jx \|^2 &\leqq \sum_{j=1}^n \| A_jE_j \|^2 \| E_jx \|^2 \\ &\leqq \max \{\| A_jE_j \|^2 : j = 1, \cdots, n\} ,\end{aligned}$$

since $\sum_{j=1}^n \| E_jx \|^2 = \| x \|^2 = 1$. Thus $\| A_1E_1 + \cdots + A_nE_n \| \leqq \max \{\| A_jE_j \|\}$. On the other hand, $\max \{\| A_jE_j \|\} \leqq \| A_1E_1 + \cdots + A_nE_n \|$, by orthogonality of $\{E_j\}$. Thus $\| \delta_1(A_1E_1 + \cdots + A_nE_n) \| = \max \{\| \bar{\delta}(A_j)E_j \|\}$. With Q_j the central carrier of E_j in $\mathfrak{A}'$,

$$\begin{aligned}\| \bar{\delta}(A_j)E_j \| &= \| \bar{\delta}(A_j)Q_j \| = \| \bar{\delta}(A_jQ_j) \| \\ &\leqq \| \bar{\delta} \| \| A_jQ_j \| = \| \bar{\delta} \| \| A_jE_j \| ,\end{aligned}$$

since the mapping $AE_j \rightarrow AQ_j$ is a $*$-isomorphism of $\mathfrak{A}^-E_j$ onto $\mathfrak{A}^-Q_j$ (see [9; Lem. 3.1.3], for example), $*$-isomorphisms between C^*-algebras are isometries [6; Cor. 6], $\bar{\delta}$ annihilates the center of $\mathfrak{A}^-$ (cf. Theorem 2), and $\bar{\delta}$ is bounded on $\mathfrak{A}^-$ (Sakai's theorem). Hence $\| \delta_1(A_1E_1 + \cdots + A_nE_n) \| \leqq \| \bar{\delta} \| \max \{\| A_jE_j \|\} = \| \bar{\delta} \| \| A_1E_1 + \cdots + A_nE_n \|$; and δ_1 is bounded. It follows that δ_1 has a bounded extension (which is a derivation) from $\mathfrak{A}_0$ to the uniform closure of $\mathfrak{A}_0$ (since by linearity, it is uniformly continuous on $\mathfrak{A}_0$) and, from Lemma 3, it has an extension $\bar{\delta}_1$ to $\mathfrak{A}_0^-$.

Now $\mathfrak{A}_0^-$ is a von Neumann algebra containing $\mathfrak{A}^-$ and $\mathfrak{C}$ (since it contains $\mathcal{P}$, the projection lattice of $\mathfrak{C}$). Thus $\mathfrak{A}_0'$ lies in $\mathfrak{A}'$ and commutes with $\mathfrak{C}$. Since $\mathfrak{C}$ is maximal abelian in $\mathfrak{A}'$, $\mathfrak{A}_0' = \mathfrak{C}$; and $\mathfrak{A}_0^-$ is of type I. From Kaplansky's theorem, $\bar{\delta}_1$ is inner. Say $\bar{\delta}_1 = \operatorname{ad} B \mid \mathfrak{A}_0^-$. Then $\delta = \operatorname{ad} B \mid \mathfrak{A}$, and $B \in \mathfrak{C}'$ since $BE - EB = \bar{\delta}_1(E) = \delta(I)E = 0$, for E in $\mathcal{P}$.

Suppose that $\mathcal{R}$ is a mixing von Neumann subalgebra of $\mathfrak{A}'$. In particular, this is the case if $\mathcal{R}$ is hyperfinite [22; Lem. 2]. With U' a unitary operator in $\mathfrak{A}'$, we note that $\operatorname{ad} U'BU'^* \mid \mathfrak{A}^- = \operatorname{ad} B \mid \mathfrak{A}^-$; for $U'BU'^*A - AU'BU'^* = U'(BA - AB)U'^* = BA - AB$ with A in $\mathfrak{A}^-$, since $BA - AB$ is in $\mathfrak{A}^-$. It follows that convex combinations $a_1U_1'BU_1'^* + \cdots + a_nU_n'BU_n'^*$ induce the same derivation as B on $\mathfrak{A}^-$, and hence that operators in the weak closure of such convex combinations induce the same derivation as $\operatorname{ad} B$ on $\mathfrak{A}^-$. Since $\mathcal{R}$ is mixing some such closure point (of convex combinations with unitary operators in $\mathcal{R}$) lies in $\mathcal{R}'$, so that δ is $\operatorname{ad} C \mid \mathfrak{A}$ for some C in $\mathcal{R}'$.

LEMMA 5. *If* ad B *induces a derivation of the* C^**-algebra* $\mathfrak{A}$, *then it induces a derivation of* $\mathfrak{A}'$. *The derivation* ad B *of* $\mathfrak{A}^-$ *is inner if and only if it induces an inner derivation of* $\mathfrak{A}'$.

PROOF. Assuming $\operatorname{ad} B$ induces a derivation of $\mathfrak{A}$, we observe, for each A in $\mathfrak{A}$ and A' in $\mathfrak{A}'$,

$$\begin{aligned}(BA' - A'B)A - A(BA' - A'B) &= BA'A - A'BA - ABA' + AA'B \\ &= (BA - AB)A' - A'(BA - AB) \\ &= 0\,,\end{aligned}$$

since $BA - AB$ lies in $\mathfrak{A}$. Thus $BA' - A'B$ lies in $\mathfrak{A}'$.

If $\operatorname{ad} B$ induces an inner derivation of $\mathfrak{A}^-$, say $\operatorname{ad} B = \operatorname{ad} C$ on $\mathfrak{A}^-$, with C in $\mathfrak{A}^-$; then $B - C$ commutes with $\mathfrak{A}^-$ and, therefore, lies in $\mathfrak{A}'$. But since C lies in $\mathfrak{A}^-$, $\operatorname{ad}(B - C) = \operatorname{ad} B$ on $\mathfrak{A}'$. Thus $\operatorname{ad} B$ induces an inner derivation of $\mathfrak{A}'$.

REMARK 6. Since the question of whether or not a derivation is inner is clearly an algebraic one; i.e., independent of the representation chosen, since all derivations of concretely represented C^*-algebras are spatial, and since each semi-finite von Neumann algebra has a faithful representation in which the commutant is finite; it would suffice to demonstrate that each derivation of a finite von Neumann algebra is inner in order to establish that each derivation of a semi-finite von Neumann algebra is inner, by virtue of the preceding lemma.

THEOREM 7. *Each derivation of a mixing von Neumann algebra is inner. In particular, each derivation of a hyperfinite von Neumann algebra is inner.*

PROOF. If $\mathcal{R}$ acting on $\mathcal{H}$ is a mixing von Neumann algebra and δ is a derivation of $\mathcal{R}$, then from Theorem 4, $\delta = \operatorname{ad} B \mid \mathcal{R}$. From Lemma 5, $\operatorname{ad} B$ induces a derivation of $\mathcal{R}'$ which is inner if and only if δ is inner. However, from Theorem 4, $\operatorname{ad} B \mid \mathcal{R}'$ is $\operatorname{ad} C \mid \mathcal{R}'$, where C may be chosen commuting with an assigned mixing von Neumann subalgebra of $\mathcal{R}''(=\mathcal{R})$. In particular C can be chosen commuting with $\mathcal{R}$. Thus $\operatorname{ad} B \mid \mathcal{R}'$ is inner as is δ.

4. Derivations by special operators

From Theorem 4, each derivation of a von Neumann algebra is the restriction to it of $\operatorname{ad} B$ for some bounded B. Under certain assumptions on B, we can take the final step and establish that this derivation is inner. We begin by noting that each of a large class of operators (a strongly dense $*$-algebra, in the case of a factor) is mobile under a von Neumann algebra.

LEMMA 8. *Each operator $A_1A_1' + \cdots + A_nA_n'$ with $A_1, \cdots, A_n$ in $\mathcal{R}$ and $A_1', \cdots, A_n'$ in $\mathcal{R}'$ is mobile under $\mathcal{R}$.*

PROOF. According to Dixmier's approximation theorem [2; Th. 1, p. 272], we can find unitary operators $U_1, \cdots, U_n$ in $\mathcal{R}$ and non-negative real numbers $\alpha_1, \cdots, \alpha_n$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\sum_{j=1}^n \alpha_j U_j A_1 U_j^*$ is close to a central operator of $\mathcal{R}$ in norm. We consider

$$\sum_{j=1}^n \alpha_j U_j(A_1A_1' + \cdots + A_nA_n')U_j^* = \sum_{j=1}^n \alpha_j U_j A_1 U_j^* A_1' + \cdots + \sum_{j=1}^n \alpha_j U_j A_n U_j^* A_n'$$

and locate $V_1, \cdots, V_m$ unitary operators in $\mathcal{R}$ and non-negative real numbers $\beta_1, \cdots, \beta_m$ with sum 1 such that $\sum_{k=1}^m \beta_k V_k(\sum_{j=1}^n \alpha_j U_j A_2 U_j^*) V_k^*$ is close to a central operator of $\mathcal{R}$ in norm. We note that $\sum_{k=1}^m \beta_k V_k(\sum_{j=1}^n \alpha_j U_j A_1 U_j^*) V_k^*$ is as close to the central operator near $\sum_{j=1}^n \alpha_j U_j A_1 U_j^*$ as $\sum_{j=1}^n \alpha_j U_j A_1 U_j^*$ is. We now consider

$$\sum_{k=1}^m \beta_k V_k(\sum_{j=1}^n \alpha_j U_j A_1 U_j^* A_1' + \cdots + \sum_{j=1}^n \alpha_j U_j A_n U_j^* A_n') V_k^*$$

and continue as before. It follows that some element of $\mathrm{co}_{\mathcal{R}}(A_1A_1' + \cdots + A_nA_n')$ is close in norm to $C_1A_1' + \cdots + C_nA_n'$ with $C_1, \cdots, C_n$ in the center of $\mathcal{R}$, this last operator lying in $\mathcal{R}'$.

From the proof of Theorem 4 (the last paragraph), we see that if $\mathrm{ad}\, B$ maps $\mathcal{R}$ into $\mathcal{R}$ and B is mobile under $\mathcal{R}'$, then $\mathrm{ad}\, B \mid \mathcal{R}$ is inner. Combining this remark with the preceding lemma, we have:

THEOREM 9. *If* $\mathrm{ad}\, B$ *maps the von Neumann algebra* $\mathcal{R}$ *into* $\mathcal{R}$ *and* $B = A_1A_1' + \cdots + A_nA_n'$, *with* A_j *in* $\mathcal{R}$ *and* A_j' *in* $\mathcal{R}', j = 1, \cdots, n$; *then* $\mathrm{ad}\, B \mid \mathcal{R}$ *is inner.*

Note that, if $\mathcal{R}$ is a factor, operators having the form described for B lie strongly dense in the algebra of all bounded operators. The theorem which follows will be subsumed in the theorem following it. However, Theorem 11 has an analytic proof and is effected by passing to groups of automorphisms, while the theorem which follows can be given a proof in terms of derivations and more algebraically. We feel that the proof and statement are of sufficient interest to give separately.

THEOREM 10. *If* $\mathrm{ad}\, B$ *induces a derivation of the von Neumann algebra* $\mathcal{R}$ *with* B *a projection, then* $\mathrm{ad}\, B \mid \mathcal{R}$ *is inner.*

PROOF. We note first that if $AC = 0$, with A and C in $\mathcal{R}$, then $ABC \in \mathcal{R}$; for $AB - BA \in \mathcal{R}$ so that $ABC - BAC = ABC \in \mathcal{R}$. From Lemma 5, $\mathrm{ad}\, B$ induces a derivation of $\mathcal{R}'$; so that $A'BC' \in \mathcal{R}'$ with $A'C' = 0$ and A', C' in $\mathcal{R}'$. Thus $0 = A'ABCC'$ for such A, A', C, C'. In particular, with E and E' projections in $\mathcal{R}$ and $\mathcal{R}'$, respectively;

$$
\begin{aligned}
0 &= EE'B(I-E')(I-E) = EE'B^2(I-E')(I-E) \\
&= EE'B[E' + (I-E')]B(I-E')(I-E) \\
&= EE'BE'B(I-E')(I-E) + EE'B(I-E')B(I-E')(I-E) \\
&= E'EB(I-E)E'B(I-E') + E'B(I-E')EB(I-E)(I-E') \\
&= 2EB(I-E)E'B(I-E') .
\end{aligned}
$$

It follows that the central carriers of $EB(I-E)$ and $E'B(I-E')$ are orthogonal for all projections E and E' in $\mathcal{R}$ and $\mathcal{R}'$, respectively.

Let Q be the union of the central carrier of $EB(I-E)$ for projections E in $\mathcal{R}$, and let P be the union of the central carriers of $E'B(I-E')$ for projections E' in $\mathcal{R}'$. Then $QP=0$, from the foregoing; and $0=PEB(I-E)=EPB(I-E)$, for each projection E in $\mathcal{R}$. Thus PB leaves $I-E$ invariant, for each such E in $\mathcal{R}$; and PB lies in $\mathcal{R}'$. Similarly $(I-P)B$ lies in $\mathcal{R}$. Now $B=PB+(I-P)B$; so that $\operatorname{ad}B \mid \mathcal{R}$ is $\operatorname{ad}(I-P)B \mid \mathcal{R}$, and $\operatorname{ad}B$ induces an inner derivation of $\mathcal{R}$.

Concerning the theorem which follows, note that, if $\operatorname{ad}B$ maps the C^*-algebra $\mathfrak{A}$ into itself, then $-(BA^* - A^*B)^*$ $(=B^*A - AB^*)$ lies in $\mathfrak{A}$, for each A in $\mathfrak{A}$; so that $\operatorname{ad}B^*$ maps $\mathfrak{A}$ into $\mathfrak{A}$. Thus each of the self-adjoint and skew-adjoint parts of B induce derivations of $\mathfrak{A}$. If each of these derivations is inner, $\operatorname{ad}B \mid \mathfrak{A}$ is inner. The question of whether or not all derivations of a von Neumann algebra are inner is reduced then to the question of whether or not spatial derivations by self-adjoint operators are. Addition of a scalar multiple of I to this operator does not affect the derivation it produces. By judicious choice of this scalar, we may arrange that our operator is positive and singular. Our next result states in essence that, if our positive singular operator annihilates a vector, the derivation to which it gives rise is inner.

THEOREM 11. *If* $\operatorname{ad}H$ *maps the von Neumann algebra* $\mathcal{R}$ *into itself,* H *is positive and* $Hx_0 = 0$ *for a vector* x_0 *such that* $[\mathcal{R}x_0]$ *has central carrier* I *in* $\mathcal{R}'$*, then* $\operatorname{ad}H \mid \mathcal{R}$ *is inner.*

PROOF. Note that $HAx_0 = (HA - AH)x_0$; so that HAx_0 is in $\mathcal{R}x_0$ when A lies in $\mathcal{R}$. Thus $HE' = E'H$, where E' is the projection (in $\mathcal{R}'$) with range $[\mathcal{R}x_0]$. Since E' has central carrier I, $A \to AE'$ is a $*$-isomorphism of $\mathcal{R}$ onto $\mathcal{R}E'$ (cf. [9, Lem. 3.1.3]). This isomorphism carries $\operatorname{ad}H \mid \mathcal{R}$ onto $\operatorname{ad}HE' \mid \mathcal{R}E'$; so that the latter is inner if and only if the former is. Now $HE' \geqq 0$, $HE'x_0 = 0$ and $[\mathcal{R}E'x_0] = E'(\mathcal{H})$. We may assume x_0 is cyclic for $\mathcal{R}$.

With t and s real, define U_{t+is} as $\exp(itH)\exp(-sH)$; so that $z \to U_z$ is an entire operator-valued function of the complex variable $z(=t+is)$. Since $Hx_0 = 0$, $U_z x_0 = x_0$ for each z. Note that $\|U_z\| = \|\exp(-sH)\|$; so that $\|U_z\| \leqq 1$ if $s \geqq 0$, since $H \geqq 0$. Note also that $A \to U_t A U_{-t}$ is an automorphism of $\mathcal{R}$ (and of $\mathcal{R}'$) since $\operatorname{ad} H$ maps $\mathcal{R}$ (and $\mathcal{R}'$) into itself. If A and A' are self-adjoint operators in $\mathcal{R}$ and $\mathcal{R}'$, respectively, then

$$\begin{aligned}(A'U_tAx_0, x_0) &= (A'U_tAU_{-t}x_0, x_0)\\ &= (U_tAU_{-t}A'x_0, x_0) = (AU_{-t}A'x_0, U_{-t}x_0)\\ &= (AU_{-t}A'x_0, x_0) = (x_0, A'U_tAx_0)\\ &= \overline{(A'U_tAx_0, x_0)}\ .\end{aligned}$$

Thus the entire function f defined by $f(z) = (A'U_zAx_0, x_0)$ is real-valued for real z. From $\|U_z\| \leqq 1$ for z in the upper half plane, we see that $|f(z)| \leqq \|Ax_0\|\,\|A'x_0\|$ for such z. From the Schwarz reflection principle, $f(\bar{z}) = \overline{f(z)}$; so that f is bounded in the entire plane. Liouville's theorem now yields that f is constant; so that

$$(U_tAx_0, A'x_0) = (Ax_0, A'x_0) = (Ax_0, U_{-t}A'U_tx_0)\ ,$$

for all real t and each self-adjoint A in $\mathcal{R}$. Since $[\mathcal{R}x_0] = \mathcal{H}$, $(U_{-t}A'U_t - A')x_0 = 0$. However, $U_{-t}A'U_t - A'$ lies in $\mathcal{R}'$. With x_0 separating for $\mathcal{R}'$, we conclude that $U_{-t}A'U_t = A'$ for all real t and each self-adjoint A' in $\mathcal{R}'$. Thus U_t lies in $\mathcal{R}$, for all real t. But iH is the norm limit of $(U_t - I)/t$ as $t \to 0$. Thus H lies in $\mathcal{R}$, and δ is inner.

REMARK 12. The argument above works equally well to show that a strongly-continuous, one-parameter unitary group with an invariant vector, and with positive spectrum which induces automorphisms of a von Neumann algebra for which the invariant vector is cyclic, consists of unitary operators in the von Neumann algebra. This is the case where H above is possibly unbounded. It is the one-dimensional analogue of the result proved in [1; see Props. 1 and 2] that the representation of the translation subgroup of the Poincaré group associated with a local quantum field theory which has a cyclic vacuum state has its image in the weak closure of the algebra of local observables of that theory. Our result could be adapted to give another proof of this fact. We are grateful to H. Araki for the privilege of seeing a pre-publication copy of [1].

5. Related results

If $\mathcal{R}$ is a finite von Neumann algebra and Tr is its center-valued trace, then, with A, B, and C in $\mathcal{R}$ and AC equal to CA we have $\operatorname{Tr}(C(BA - AB)) = 0$. Thus if the derivation δ of $\mathcal{R}$ is to be inner and C commutes with A, we should have $\operatorname{Tr}(C\delta(A)) = 0$. As further evidence that derivations of von Neumann algebras are inner, we prove:

THEOREM 13. *If δ is a derivation of the finite von Neumann algebra $\mathcal{R}$ and A and C in $\mathcal{R}$ commute and A is self-adjoint, then* $\operatorname{Tr}(C\delta(A)) = 0$. *In* *particular*, $\mathrm{Tr}(\delta(A)) = 0$, *for each* A *in* $\mathcal{R}$.

PROOF. From Theorem 4, $\delta = \mathrm{ad}\, B \mid \mathcal{R}$, for some bounded operator B. Recall that if T and S are in $\mathcal{R}$ and $TS = 0$, then $TBS = T(BS - SB)$ lies in $\mathcal{R}$. In particular, with E, F orthogonal projections in $\mathcal{R}$, EBF lies in $\mathcal{R}$.

Let $\mathcal{A}$ be a (self-adjoint) maximal abelian subalgebra of $\mathcal{R}$ containing A. In [19; Ch. II], von Neumann introduces the concept of a "diagonal part" of an operator in $\mathcal{R}$ relative to $\mathcal{A}$. In [11, Lem. 1] it is shown that a diagonal process $\mathcal{D}$ (not unique in general) exists which maps each bounded operator B onto an operator $\mathcal{D}(B)$ in $\mathcal{A}'$ which is a weak limit point of operators $B^{|E_1|\cdots|E_n}$ with $E_1, \cdots, E_n$ projections in $\mathcal{A}$, where $B^{|E} = EBE + (I - E)B(I - E)$. Thus $B - \mathcal{D}(B)$ is a weak limit point of operators $B - B^{|E_1|\cdots|E_n}$. But $B^{|E_1|\cdots|E_n} = \sum_{j=1}^{m} F_j B F_j$ with $\{F_j\}$ a family of mutually orthogonal projections in $\mathcal{A}$ having sum I (since $E_1, \cdots, E_n$ commute). Hence $B - B^{|E_1|\cdots|E_n} = \sum_{j \neq k} F_j B F_k$. With B giving rise to a derivation, each term of this sum lies in $\mathcal{R}$, as noted in the first part of the proof. Thus $B - \mathcal{D}(B)$, a weak limit point of operators in $\mathcal{R}$, lies in $\mathcal{R}$. Since $\mathcal{D}(B)$ lies in $\mathcal{A}'$, $\mathrm{ad}\, B$ and $\mathrm{ad}(B - \mathcal{D}(B))$ are the same on $\mathcal{A}$. Thus

$$\mathrm{Tr}\,(C\delta(A)) = \mathrm{Tr}\,(C\,\mathrm{ad}\,B(A)) = \mathrm{Tr}\,(C[\mathrm{ad}\,(B - \mathcal{D}(B))](A)) = 0\ ,$$

by the remarks preceding this theorem.

REMARK 14. In [5] the existence of maximal hyperfinite subfactors of a factor of type II_1 is established. One could extend the argument to show that their relative commutant is commutative. It may well consist of scalars in all cases, though this is not known. At any rate, examples of hyperfinite subfactors whose relative commutant consists of scalars are known (making use of the group measure space examples of [16; pp. 192–209] and the fact that, if the group is commutative, the resulting factor is hyperfinite [18; Lem. 5.2.3, 4; Cor. 4.1]). Let $\mathcal{M}'$ be of type II_1 and $\mathcal{M}_0'$ a hyperfinite subfactor with relative commutant scalars. If δ is a derivation of $\mathcal{M}$, we can choose B so that $\mathrm{ad}\, B \mid \mathcal{M} = \delta$ and B is in $\mathcal{M}_0$. Now the relative commutant of $\mathcal{M}_0'$ in $\mathcal{M}'$ is the relative commutant of $\mathcal{M}$ in $\mathcal{M}_0$. If a subfactor of a factor has an abelian subalgebra which is maximal abelian in the larger factor (many examples of this exist) it will have relative commutant the scalars. The converse of this may well hold; viz., if a subfactor of a factor has relative commutant the scalars, then some (maximal) abelian subalgebra of it is maximal abelian in the larger factor. If this does hold, say $\mathcal{A}$ in $\mathcal{M}$ is maximal abelian in $\mathcal{M}_0$, then a diagonal part $\mathcal{D}(B)$ of B relative to $\mathcal{A}$ lies in $\mathcal{A}$, and hence in $\mathcal{M}$. With B giving rise to a derivation of $\mathcal{M}$, $B - \mathcal{D}(B)$ lies in $\mathcal{M}$ (from the proof of the preceding theorem) as does $\mathcal{D}(B)$; so that B lies in $\mathcal{M}$ and δ is inner.

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(Received March 3, 1965)

Added June 14, 1965. Since the preceding results were obtained, several further facts related to derivations have been established. J. Ringrose and the author [28] proved that each derivation of the von Neumann group algebra of a countable discrete group is inner. S. Sakai [30] then completed the arguments of the present paper to obtain:

Theorem 15. *Each derivation of a von Neumann algebra is inner.*

J. Ringrose and the author [29] gave a simplified proof of this result, using a device of Sakai's, in a paper which derives the corollary that each norm-continuous representation of a connected Lie group by $*$-automorphisms of a von Neumann algebra is a representation by inner automorphisms. Reducing to this result, H. Borchers [27] showed that each strongly-continuous, one-parameter unitary group with spectrum bounded below, which induces automorphisms of a von Neumann algebra, induces inner automorphisms. With this same reduction, G. Dell' Antonio (private communication) showed that a weakly-continuous, one-parameter group of $*$-automorphisms of a von Neumann algebra is induced by a strongly-continuous, one-parameter unitary group in the algebra if it satisfies a certain condition akin to the semi-boundedness of the spectrum. (These last results state, roughly, that the energy and momentum of a quantum field are observable without the assumption of a vacuum state.)

Since proving Theorem 4, we have felt that the step to Theorem 15 should be a straightforward matter of showing that $\overline{\text{co}}_{\mathcal{R}'}(B)$ contains an operator in $\mathcal{R}$, when B induces a derivation of $\mathcal{R}$. The present addendum is prompted by finding such a proof. Though we give the proof for an arbitrary von Neumann algebra rather than for factors alone, the essential ideas are found in the latter case. In sketch, Zorn's Lemma yields a minimal, non-null, convex, weak-operator compact subset $\mathcal{K}$ of $\overline{\text{co}}_{\mathcal{R}'}(B)$ invariant under unitary operators in $\mathcal{R}'$. By minimality, the elements of $\mathcal{K}$ have the same norm. But if B_1 and B_2 are distinct elements of $\mathcal{K}$, $\overline{\text{co}}_{\mathcal{R}'}(B_1 - B_2)$ contains some aI, $a \neq 0$ (this last is somewhat over-simplified). Thus $B_3 - B_4 = bI$ with $b > 0$ and B_3, B_4 (positive) operators in $\mathcal{K}$; so that $\|B_3\| > \|B_4\|$. Hence $\mathcal{K}$ consists of a single operator which, by invariance, lies in $\mathcal{R}$.

PROOF OF THEOREM 15. From Theorem 4, our derivation has the form $\text{ad}\, B \mid \mathcal{R}$; and from the discussion preceding Theorem 11, we may assume that $B \geqq 0$. From Theorem 2, B is in $\mathcal{C}'$, where $\mathcal{C}$ is the center of $\mathcal{R}$. Thus if $\{Q_\alpha\}$ is an orthogonal family of projections in $\mathcal{C}$ with sum I such that $\text{ad}\, B \mid \mathcal{R}Q_\alpha = \text{ad}\, A_\alpha \mid \mathcal{R}Q_\alpha$, where A_α is in $\mathcal{R}Q_\alpha$ and $\sup_\alpha \{\|A_\alpha\|\} < \infty$; then $\text{ad}\, B \mid \mathcal{R} = \text{ad}\, A \mid \mathcal{R}$, where $A = \sum A_\alpha$ is in $\mathcal{R}$. Choosing Q_α cyclic under $\mathcal{C}'$, it will suffice to establish the result (with uniform bound) for $\mathcal{C}$ countably decomposable. In this case, choosing E' a cyclic projection in $\mathcal{R}'$ with central carrier I [9; Lemma 3.3.1] and passing to the faithful representation $\mathcal{R}E'$ of $\mathcal{R}$ on $E'(\mathcal{H})$ (with commutant $E'\mathcal{R}'E'$), we can assume that $\mathcal{R}'$ is countably decomposable. (The bounds are not increased by this reduction.) Finally, using the central portions of $\mathcal{R}'$ corresponding to pure type and the process just described, we can assume that $\mathcal{R}'$ is either of finite type or of type III and countably decomposable.

For brevity, we say that a set of operators stable under the mappings $A \to A^{|E}$, $A \to U^*AU$, with E a projection and U a unitary operator in $\mathcal{R}'$ is 'stable'. The positive operators in the closed ball of radius $\|B\|$ with center 0 is a convex, weak-operator closed (in fact, compact) stable set containing B as is $B + \mathcal{R}'$. Let $\mathcal{K}(B)$ be the intersection of all such sets (one could show that $\mathcal{K}(B) = \overline{\mathrm{co}}_{\mathcal{R}'}(B)$). Zorn's lemma yields the existence of a set $\mathcal{K}$ minimal with respect to inclusion among the non-null, convex, compact stable subsets of $\mathcal{K}(B)$. If P is a projection in $\mathcal{C}$ and B_1 is in $\mathcal{K}$, $\{B_2 : \|B_2P\| \leqq \|B_1P\|, B_2 \text{ in } \mathcal{K}\}$ is such a subset of $\mathcal{K}$; so that $\|B_2P\| = \|B_1P\|$ for each B_2 in $\mathcal{K}$.

We prove that $\mathcal{K}$ consists of a single operator, which must lie in $\mathcal{R}$ by stability; by showing that the set $\mathcal{K}_0$ of differences of pairs of operators in $\mathcal{K}$ contains only 0. Note that $\mathcal{K}_0$ is a convex, compact, stable set of self-adjoint operators in $\mathcal{R}'$ (since $B + \mathcal{R}'$ contains $\mathcal{K}$). If B_0 is a non-zero operator in $\mathcal{K}_0$, using $-B_0$, if necessary, we may assume that $B_0^+ \neq 0$, where B_0^+ and B_0^- are the positive and negative *parts* of B_0, respectively. If $C_{B_0^+}C_{B_0^-} = 0$ then $\overline{\mathrm{co}}_{\mathcal{R}'}(B_0)$ contains some $C_1 + B_0^-$, with $C_1C_{B_0^+} = C_1$, $C_1 > 0$ and C_1 in $\mathcal{C}$. (Apply the Dixmier process [2; Ch. III § 5, 31; XXII p. 3.33 Lemma 15, 29; Lemma 2] to B_0^+ in $\mathcal{R}'C_{B_0^+}$ extending the unitary operators as $I - C_{B_0^+}$.) Since $\mathcal{K}_0$ is stable, $C_1 + B_0^- = B_1 - B_2$, with B_1, B_2 in $\mathcal{K}$. For an appropriate central projection P and some positive a, $B_1P - B_2P = C_1P > aP$. But then $\|B_1P\| \geqq \|B_2P + aP\| = \|B_2P\| + a > \|B_2P\|$ (recall that operators in $\mathcal{K}$ are positive), contradicting a property of $\mathcal{K}$.

We may assume that $C_{B_0^+}C_{B_0^-} \neq 0$. In this case, with E the range projection of B_0^+, $\overline{\mathrm{co}}_{E\mathcal{R}'E}(B_0E)$ contains some non-zero C_1E, where $C_1 > 0$ and C_1 is in $\mathcal{C}C_{B_0^+}$; and $\overline{\mathrm{co}}_{(I-E)\mathcal{R}'(I-E)}(B_0(I-E))$ contains some $C_2(I-E)$ where $C_2 < 0$, C_2 is in $\mathcal{C}C_{B_0^-}$ and $C_1C_2 \neq 0$. Thus $\mathcal{K}_0$ contains $C_1E + C_2(I-E) = B_1 - B_2$, with B_1 and B_2 in $\mathcal{K}$. Now $B_1^{|E} - B_2^{|E} = C_1E + C_2(I-E)$, and $B_1^{|E}$, $B_2^{|E}$ are in $\mathcal{K}$. We may assume that B_1 and B_2 commute with E. Since $\mathcal{K}$ is convex and contains B_2 and $(B_2 + C_1)E + (B_2 + C_2)(I-E)$, it contains $(B_2 + tC_1)E + (B_2 + tC_2)(I-E)$ $(=A_t)$, for each t in $[0, 1]$. For some projection P in $\mathcal{C}$, $aP < C_1P$ and $C_2P < bP$, with a and $-b$ positive numbers. Then $\|A_tP\| = \max\{\|(B_2 + tC_1)PE\|, \|(B_2 + tC_2)P(I-E)\|\}$; and, for small t, $\|(B_2 + tC_1)PE\| \geqq \|B_2PE\| + ta$, $tb + \|B_2P(I-E)\| \geqq \|(B_2 + tC_2)P(I-E)\|$. If $\|B_2PE\| \geqq \|B_2P(I-E)\|$ then for small t, $\|A_tP\| > \|B_2P\|$. If $\|B_2P(I-E)\| > \|B_2PE\|$, then for very small t, $\|B_2P(I-E)\| > \|(B_2 + tC_2)P(I-E)\| \geqq \|(B_2 + tC_1)PE\|$; so that $\|B_2P\| > \|A_tP\|$ for such t. In any event, we have operators A_t and B_2 in $\mathcal{K}$ with $\|A_tP\| \neq \|B_2P\|$.

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